Bending Stresses

Assumptions:
1. Material is homogeneous and isotropic.
2. The beam is initially straight and unstressed.
3. Elastic limit is not exceeded.
4. Young's Modulus is same in compression and tension.
5. The plane sections of beam remains plain before and after bending.

Pure bending / Simple bending:

A member is said to be in pure bending when it is subjected to equal and opposite couples in the plane along the longitudinal axis. For pure bending, the shear force is zero.

\[ S.F. = 0 \quad \text{pure bending} \]

\[ \rightarrow \text{Not the case of pure bending.} \]

\[ S.F. \neq 0 \]
Fig: A deformed beam loaded transversely to its axis.

Note:
It's not necessary for the complete beam to undergo pure bending part of the beam may also undergo pure bending.
Neutral Surface: It is the surface which experiences neither tension nor compression.
- There is only one neutral surface.

Neutral axis: It is the line of intersection of cross-section with the neutral surface, there can be any no. of infinite neutral axis.
Normal strain in $ab = \varepsilon = \frac{a'b'-ab}{ab}$

$$\varepsilon = \frac{(R-y)d\theta - R\theta}{Rd\theta}$$

$$\varepsilon = -\frac{y}{R}$$

From this equation it is observed that the normal strain is directly proportional to the distance from the neutral axis ($y$).
In the above eqn. -ve sign shows that for +ve Bending Moment (Sagging) there will be compressive stresses when \( y \) is +ve, i.e., above the neutral axis. Similarly, when \( y \) is -ve (below N.A.) strain is +ve i.e., tensile.

As the loading is within proportionality limit hooks low can be apply. i.e. \( \sigma \propto \varepsilon \)

\[
\sigma = E \varepsilon
\]

\[
\sigma = E \left( \frac{-y}{R} \right)
\]

\[
\frac{\sigma}{E} = \frac{-y}{R}
\]

or

\[
\frac{\sigma}{E} = \frac{y}{R}
\]

\( \sigma = \) bending stress

\( E = \) Young's modulus

\( y = \) distance from N.A.

\( R = \) Radius of curvature.

**Equilibrium:**

As there is no external force applied in axial direction therefore net force in axial direction must be zero.
Force equilibrium:

\[ \sigma = -\frac{E y}{R} \]

\[ dF = \sigma dA \]

\[ dF = -\frac{E}{R} y dA \]

\[ F = -E \int \gamma dA = 0 \]

\[ \int \gamma dA = A \cdot \bar{y} \]

\[ A \neq 0 ; \quad \bar{y} = 0 \]

\( \bar{y} = 0 \) shows that the centroidal axis coincides with N.A. therefore in pure bending centroidal axis always coincides with neutral axis.

Moment equilibrium:

\[ \sigma = -\frac{E y}{R} \]

\[ dF = \sigma dA \]

\[ dF = -\frac{E}{R} y dA \]

Moment = \( dF \times y \)

Total Moment (M) = \( \int dF \times y \)

\[ M = \int -\frac{E}{R} y dA \cdot y \]
\[ M = - \frac{E}{R} \int y^2 \, dA \]

\( \int y^2 \, dA \) is the Second moment of area about N.A. and this is known as area moment of inertia. If it is designated by \( I \).

\[ \int y^2 \, dA = I_{zz} \]

\[ M = - \frac{E}{R} I \]

Let \( M_R \) be the resisting moment

\[ M = - M_R \]

\[ \Rightarrow M_R = \frac{EI}{R} \]

\[ M_R = \frac{EI}{R} \Rightarrow \frac{M_R}{I} = \frac{E}{R} \]  \( \text{(2)} \)

From eqn 1 & 2.

\[ \frac{E}{I} = \frac{M_R}{I} \]

\[ \frac{E}{I} = \frac{M_R}{I} \]

Moment of resistance

Euler-bernoulli bending equation.
Economical Sections:

In a beam having rectangular or circular section the fibers near neutral axis are understressed (less) compared with those at the top or bottom. The fact that the large portion of the cross-section is thus understressed makes it inefficient for resisting flexure or bending.

The expression \( \frac{M}{I} = \frac{5}{y} \) indicates that if the area of the beam of rectangular cross-section is rearranged (redistributed) so as to maintain same depth & same area the MOI would be greatly increased resulting in greater moment carrying capacity.

This moment resisting capacity is due to placing more material at greater distance from the N.A.

In order to obtain the max. resistance to bending it is advisable to use section which have large area away from the N.A. and hence I-Sections & T-Sections are preferable.
Increasing MOI

\[ I > T > \square > \bigcirc > \triangle \]

Increase in resistance to bending

Resistance to bending
More

Resistance to bending less

\textbf{Section Modulus (Z):}

\[ \frac{I}{Y_{\text{max}}} \]

is known as Section Modulus.

If Section Modulus (Z) is more, \( \sigma_{\text{max}} \) will be less and the resistance to the bending is more & hence chances of bending failure will be less.

\textbf{Note:}

In case of axial loading, c/s area is considered where as in the case of bending, Section Modulus is considered.

\[ \sigma = \frac{M Y}{I} \]

\[ \sigma_{\text{max}} = \frac{M Y_{\text{max}}}{I} \]

\[ \frac{I}{Y_{\text{max}}} = Z \quad ; \quad \sigma_{\text{max}} = \frac{M}{Z} \quad ; \quad M = \sigma_{\text{max}} \cdot Z \]
**Beams of Uniform Strength:**

A beam is said to be beam of uniform strength if the max. Bending stress is same at each & every section.

**Case-1:** Cantilever beam subjected to moment \( M \) at the free end.
\[ \sigma_{\text{max}} = \frac{M \cdot Y_{\text{max}}}{I} = \frac{M \cdot (d/2)}{bd^3} \]

\[ \tau_{\text{max}} = \frac{6M}{bd^2} \]

As \( \sigma_{\text{max}} \) is independent of \( x \), therefore \( \tau_{\text{max}} \) is constant at each and every section, therefore, this is the beam of constant strength.

Generally beams are subjected to transverse loading and hence B.M. changes along the length of the beam so it is not a beam of constant strength, to make such a beam a beam of constant strength, two techniques are followed.

[a] Varying width, keeping the depth constant.
[b] Varying depth, keeping the width constant.
\[(\sigma_{\text{max}})_{1-1} = \frac{6Px}{bXd^2}\]

\[\sigma_{\text{max}}|_{2-2} = \frac{6PL}{b_Ld^2}\]

**Case 1**: Varying width, keeping depth constant

For a uniform strength beam

\[\sigma_{\text{max}}|_{1-1} = \sigma_{\text{max}}|_{2-2}\]

\[\frac{6Px}{bXd^2} = \frac{6PL}{b_Ld^2}\]

\[b_X = b_L\left(\frac{x}{L}\right)\]

Let width at fixed end \(b_L = b\)

\[b_X = b\left(\frac{x}{L}\right)\]

![Diagram of a beam with varying width and depth]
**Case 2:** Keeping the width constant, varying depth.

\[ \sigma_{\text{max}}|_{1-1} = \sigma_{\text{max}}|_{2-2} \]

\[ \frac{6P\alpha}{bd_x^2} = \frac{6PL}{bd_L^2} \]

\[ d_x = d_L \sqrt{\frac{\alpha}{L}} \]

Let the depth at the fixed end be 'd'

\[ d_L = d \]

\[ d_x = d \sqrt{\frac{\alpha}{L}} \]

\[ d_x \propto \sqrt{\alpha} \]

**Important observation:**

Neutral surface

Trapezoidal c/s

Before bending

(Rectangular)

After bending

L \uparrow \text{Width} \uparrow
[1] A

[2] A

\[ \tau = \frac{VAY}{Ib} \]  

pure bending  \( \nu = 0 \)

\( \tau = 0 \)
\( \tau = GY; \ Y = 0 \)

[3]

\[ E = 200 \text{GPa} \]
\[ R' = 500 + \frac{5}{2} = 502.5 \text{mm} \]
\[ I = \frac{\pi}{64} (5)^4 \]

\[ \frac{E}{I} = \frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R} \]

\[ = M = 200 \times 10^3 \times \frac{\pi}{64} (5)^4 \]
\[ \frac{N \cdot \text{mm}}{502.5} \]

\[ \Rightarrow 12210.79 \text{N-mm} \]
\[ \Rightarrow 12.210 \text{N-m.} \]

[4] A

[5]
Chapter 4

BENDING STRESSES

01. In the flexure theory of beams the neutral axis has the following characteristic.
   (A) Always passes through the centroid of the cross section
   (B) Always remains straight after bending
   (C) Always lies at the mid height of cross-section
   (D) Longitudinal stress is maximum along the axis.

02. In a prismatic beam under the action of pure bending:
   (A) Both the shear stress and shear strain are zero
   (B) Shear stress is zero and shear strain is non-zero.
   (C) Shear stress is non-zero and shear strain is zero.
   (D) Both shear stress and shear strain are non-zero.

03. A steel wire of diameter 5mm is bent around a cylindrical drum of radius 0.5m. The steel wire has modulus of elasticity of 200 GPa. Find the bending moment in the wire in N-m.

\[
R = 0.5 \text{ m} \quad \text{d} = 5 \text{mm}
\]

04. A steel beam replaced by a corresponding aluminium beam of same cross-sectional shape and dimensions, and is subjected to same loading. The maximum bending stress will
   (a) be unaltered  (b) increase
   (c) decrease     (d) vary in proportion to their modulus of elasticity.
05. A homogenous prismatic simply supported beam is subjected to a point load $F$. The load can be placed anywhere along the span of the beam. The very maximum flexural stress developed in the beam is

$$\frac{3FL}{2BD^2}$$

06. A timber beam of 100 mm width and 200 mm depth is reinforced with two steel plates of 100 mm width and 5 mm thickness as shown in figure I and figure II. Which one of the following statements is correct for the same value of bending stress in the timber?

(a) Moment of resistance in figure I will be more than that in figure II
(b) Moment of resistance in figure II will be more than that in figure I
(c) Moment of resistance in figure I will be equal to that in figure II
(d) No logical comparison can be made
07. A hollow circular shaft of inside diameter 10 mm and outside diameter 20 mm is subjected to a pure symmetric - bending moment of 200 N-m. The magnitude of bending stress at a point in the plane of loading, which is at a distance of 5 mm from the neutral axis, is (A) 0 (B) 68.8 MPa (C) 135.8 MPa (D) 271.6 MPa

08. A test is conducted on a beam loaded by end couples. The fibers at layer CD are found to lengthen by 0.03 mm and fibers at layer AB shorten by 0.09 mm is 20 mm gauge length as shown in the figure. Taking \( E = 2 \times 10^6 \) N/mm², the flexural stress at the top fibre in N/mm² is

09. A beam of flexural rigidity 8x10⁶ Nm² is subjected to four point bending as shown in figure. The radius of curvature of the portion BC of the beam is

(A) 800 m  (B) 600 m  (C) 1000 m  (D) continuously variable
10. Statement (I): A simply supported beam AB of constant EI throughout, when subjected to pure terminal couples as shown in figure, will bend into an arc of a circle.
Statement (II): Theory of simple bending establishes relationships from among M, f, R, y, E and I

11. A structural steel beam has an unsymmetrical I-cross-section. The overall depth of the beam is 200 mm. The flange stresses at the top and bottom are 120 N/mm² and 80 N/mm², respectively. The depth of the neutral axis from the top of the beam will be
   (a) 120 mm  (b) 100 mm  (c) 80 mm  (d) 60 mm

12. A square beam laid flat is then rotated in such a way that one of its diagonal becomes horizontal. How is its moment capacity affected?
   (a) Increases by 41.4%  (b) Increases by 29.27%
   (c) Decreases by 29.27%  (d) Decreases by 41.4%

13. A thin steel ruler having its cross-section of 0.0625 cm x 2.5 cm bent by couples applied at its ends so that its length is equal to 25 cm. When bent, as a circular arc, subtends a central angle $\theta = 60^\circ$. Take $E = 2 \times 10^5$ kg/cm². The maximum stress induced in the ruler and the magnitude is
   (a) 2618 kg/cm²  (b) 2512 kg/cm²  (c) 2406 kg/cm²  (d) 2301 kg/cm²

14. A beam of rectangular cross-section is made of two different materials, as shown in figure. The location of the neutral axis is at

   (A) $z = 2a$  (B) $z = 1.5a$  (C) $z = 1.3a$  (D) $z = 0.75a$
15. An unspecified pure bending moment is used to bend an aluminium rod of radius 2.5 mm elastically into a circular ring of radius 2 m. If the same bending moment is used to bend elastically a copper rod of radius 2 mm, the radius of the resulting ring (in m) is (elastic modulus of aluminium is 70 GPa and elastic modulus of copper is 120 GPa)
   (A) 0.702    (B) 1.404    (C) 1.755    (D) 2.808

16. The structure shown below is of rectangular cross-section and carries a load of 10kN at its free end E. Maximum bending stress (in MPa) developed in the beam due to the external load is ___.

![Beam Diagram]

The depth of the beam is 300mm and the width is 150mm.

17. In a beam of uniform strength the extreme fibers at every cross-section are stressed to the maximum allowable stress. Consider a solid circular beam of uniform strength subjected to bending moment. In this beam, the diameter of the cross-section at any section is proportional
   (A) to cube root of the bending moment at that section
   (B) to the square root of the bending moment at that section
   (C) to the bending moment at that section
   (D) inversely to the bending moment at that section

18. A cantilever beam of T-section, shown in figure is carrying a couple moment $M_0$ at the free end. Maximum magnitude of bending stress will occur at

![T-section Diagram]
19. A cantilever beam of length L, supports a concentrated load P at the free end. The cross-section of the beam is rectangular with constant width b and varying depth h. The depth h of this idealized cantilever beam varies in such a way that the maximum normal stress at every cross-section remains equal to the allowable bending stress. Considering only the bending stresses, the depth h, of the fully stressed beam at any distance x from the free end shall vary

\[ h = \text{(A) with square of } x \quad \text{(B) with square root of } x \quad \text{(C) linearly with } x \quad \text{(D) with cube of } x \]

**Linked Data Questions 20 and 21**

A massless beam has a loading pattern as shown in the figure. The beam is of rectangular cross-section with a width of 30 mm and height of 100 mm.

20. The maximum bending moment occurs at
   (a) Location B       (b) 2675 mm to the right of A
   (c) 2500 mm to the right of A (d) 3225 mm to the right of A

21. The maximum magnitude of bending stress (in MPa) is given by
   (a) 60.0           (b) 67.5          (c) 200.0          (d) 225.0
\( M = \frac{Pab}{L} = \frac{FX(L-x)}{L} \)

For \( M_{\text{max}} \), \( \frac{dM}{dx} = 0 \) \( \frac{F}{L} (L-2x) = 0 \)

\( x = \frac{L}{2} \)

\( M_{\text{max}} = \frac{FL/2 (L-L/2)}{L} = \frac{FL}{4} \)

\( \Delta_{\text{max}} = \frac{MY}{I} = \frac{FL}{4} \cdot \frac{P/2}{BD^2/12} = \frac{3FL}{2BD^2} \)

\[ \text{[7]} \]

\( M = 200 \text{ N-m} \)

\( \Delta_{Y=5\text{mm}} = \frac{MY}{I} \)

\( \Rightarrow \frac{200 \times 5}{1000 \times \frac{\pi}{64} (0.020^4 - 0.010^4)} \)

\( \Rightarrow 135.8122 \text{ MPa} \)

\[ \text{[8]} \]

\( \delta_{CD} = 0.03 \text{ mm} \)

\( \delta_{AB} = 0.09 \text{ mm} \)

\( L_{CD} = L_{AB} = 20 \text{ mm} \)
\[ \epsilon_{AB} = \frac{0.03}{20} = 1.5 \times 10^{-3} \]
\[ \epsilon_{CD} = \frac{0.09}{20} = 4.5 \times 10^{-3} \]

\[ \frac{\epsilon_{CD}}{x} = \frac{\epsilon_{AB}}{100-x} \]
\[ \frac{1.5 \times 10^{-3}}{x} = \frac{4.5 \times 10^{-3}}{100-x} \]
\[ 3x = 100-x \]
\[ x = 25 \text{ mm} \]

\[ (\text{top fibre}) = \frac{MV_{\text{top}}}{I} = \frac{E}{R} \]

\[ (\text{top fibre}) = ? \]
\[ \sigma_{CD} = \epsilon_{CD} \times E \]
\[ = 1.5 \times 10^{-3} \times 2 \times 10^5 \text{ N/mm}^2 \]
\[ = 300 \text{ MPa} \]

\[ \sigma_{\text{top}} = \frac{12}{175} = \frac{200}{25} \text{ (from Similair \( \Delta \) Rule)} \]

\[ \sigma_{\text{top fibre}} = 900 \text{ MPa} \]
Bc region is pure bending region. So Bending Moment in Bc region is constant & SF. = 0;

\[ P = 100 \times 1 = 100 \text{ N} \]

\[
\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}
\]

\[ R = \frac{EI}{M} = \frac{8 \times 10^4}{100} = 800 \text{ m} \]

\( \sigma_{\text{top}} = 120 \text{ N/mm}^2 \)

\( \sigma_{\text{bottom}} = 80 \text{ N/mm}^2 \)

\( x = 200 - x \)

\[ \sigma = \frac{My}{I} \]

\[ \sigma \propto y \]

\[ \frac{12 \phi}{8 \phi} = \frac{x}{200 - x} \]

\[ = \frac{3}{2} = \frac{x}{200 - x} \]

\[ 600 - 3x = 2x \]

\[ 5x = 600 \]

\[ x = \frac{600}{5} = 120 \text{ mm} \]
12.

\[ I = \frac{a^4}{12} \]
\[ Z_1 = \frac{a^3}{6} \]
\[ Z_2 = \frac{a^4}{12} \cdot \frac{a}{\sqrt{2}} = \frac{a^3}{6\sqrt{2}} \]

\[ Z_1 > Z_2 \]

Moment capacity \( A_{f} = \frac{Z_1 - Z_2}{Z_1} = \frac{a^3/6 - a^3/6\sqrt{2}}{a^3/6} \)

\[ \Rightarrow \frac{1 - \frac{1}{\sqrt{2}}}{1} = 0.2928 \]

\[ \text{by } 29.28\% \]

13.

\[ E = 2 \times 10^6 \text{ kg/cm}^2 \]

\[ \theta = 60^\circ \]
$L = R \theta$

$\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$

$\sigma = \frac{Ey}{R}$

$R = \frac{L}{\frac{\theta}{3}} = \frac{25}{\frac{\theta}{3}}$

$(R+y) \theta = L$

$\Rightarrow$

\[ y_{max} = \frac{0.0625}{2} = 0.03125 \text{ cm} \]

$\sigma = \frac{2 \times 10^6 \times 0.03125}{25 \times 3 \pi} = 261.799 \text{ kg/cm}^2$

**Copper rod.**

Radius = 2 mm

$R_{\text{ring}} = \ ?$

$E_{\text{Cu}} = 120 \text{ GPa}$

\[ \frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R} \]

$\Rightarrow M = \left( \frac{EI}{R} \right)_{\text{Al}} = \left( \frac{EI}{R} \right)_{\text{Cu}}$

$\Rightarrow \frac{70 \times \frac{\pi}{64} (2.5)^4}{2} = \frac{120 \times \frac{\pi}{64} (2)^4}{(R_{\text{Cu}})_{\text{ring}}}$

$(R_{\text{Cu}})_{\text{ring}} = 1.404 \text{ m}$
The diagram shows a beam with various forces and moments acting on it. The calculations are as follows:

\[ M_0 = 50 - 10 = 40 \text{ kN.m} \]

\[ d = 300 \text{ mm} \]

\[ b = 150 \text{ mm} \]

\[ \sigma = \frac{40 \times 10^3 \text{ N.m} \times 150 \times 10^3}{\frac{b}{150} \times 300^3} = 17.77 \text{ MPa} \]

For a uniform strength beam, the maximum bending stress is the same at every section.

\[ \sigma = \frac{MY}{I} \]

\[ M \Rightarrow \frac{\sigma \times I}{Y} \Rightarrow \frac{\sigma \times \pi d^4}{64} \]

\[ M = \frac{\sigma \times \pi d^3}{32} \]

\[ M \alpha d^3 \quad \left[ \sigma = \text{const} \right] \]

\[ d \alpha 3\sqrt{M} \]
\[ \tau_{\text{max}} \text{ will occur at pixel } P \]

bending moment same but \( y \) should be max.

\[ y_i > y_2 \]

bottom fibre bending stress is maximum throughout.

\[ \tau = \frac{M \cdot y_{\text{max}}}{I} \]

---

\[ R_A + R_B = 6000 \text{ N} \]

\[ \Sigma M_A = 0 \]

\[ R_B \times 4 = 6000 \times 3 \]

\[ V_X = R_A - 3000(x-2) \]

\[ \Rightarrow 1500 - 3000x + 6000 = 0 \]

\[ \Rightarrow 7500 = 3000x \]

\[ x = \frac{7500}{3000} = 2.5 \text{ m} = 2500 \text{ mm from A, SF=0, BM-max.} \]
\[ M_x = +R_A x - 3000 \left( x - 2 \right)^2 \]

at \( x = 2.5 \text{ m} \) bending moment is max.

\[ M = +1500 \times 2.5 - 3000 \left( \frac{0.5}{2} \right)^2 \]

\[ = 3750 - 375 \Rightarrow 3375 \text{ N-m}. \]

\[ \sigma_{\text{max}} = \frac{3375 \times 10^3 (N-mm) \times 50}{30 \times 100^3} \Rightarrow 67.5 \text{ MPa}. \]